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A Bayesian Approach to Portfolio Selection in Multicriteria Group Decision Making

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Abstract

In the a-posteriori approach to multicriteria decision making the idea is to first find a set of interesting (usually non-dominated) decision alternatives and then let the decision maker select among these.
Often an additional demand is to limit the size of alternatives to a small number of solutions. In this case, it is important to state preferences on sets. In previous work it has been shown that independent normalization of objective functions (using for instance desirability functions) combined with the hypervolume indicator can be used to formulate such set-preferences.
A procedure to compute and to maximize the probability that a set of solutions contains at least one satisfactory solution is established. Moreover, we extend the model to the scenario of multiple decision makers. For this we compute the probability that at least one solution in a given set satisfies all decision makers. First, the information required a-priori from the decision makers is considered. Then, a computational procedure to compute the probability for a single set to contain a solution, which is acceptable to all decision makers, is introduced. Thereafter, we discuss how the computational effort can be reduced and how the measure can be maximized. Practical examples for using this in database queries will be discussed, in order to show how this approach relates to applications.

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1. Introduction

Here we discuss scenarios for multicriteria decision making that occur in situations where multiple decision makers (DMs) are involved. The general idea is that first all DMs state and normalize their objectives. Then a small number of alternative solutions is computed. These solutions are discussed in a decision making session where all DMs meet with the goal to find a solution that is acceptable (satisfactory) for all of them.

We will introduce the main symbols that will occur in the discussion below. They will be considered in more detail in the text that follows.

Nomenclature

DM	decision maker
X	search space
n	size of search space (if finite space is considered)
$x \in X$	solution alternative
S	portfolio (subset of X of size k)
$f_1, \dots, f_m$	objective functions or criteria $f_i: X \to \mathbb{R}$ (to be maximized)
$D_1^i, \dots, D_m^i$	desirability functions of i-th DM : $D_j^i: \mathbb{R} \to [0,1]$, $1 \le j \le m, 1 \le i \le q$
$D_1, \dots, D_m$	desirability functions if only a single DM is considered $D_i : \mathbb{R} \to [0,1], 1 \le i \le m$
D	aggregation of desirability functions or *desirability index*: $D: [0,1]^m \to [0,1]$
$P_1^i, \dots, P_m^i$	acceptance probability functions of i-th DM : $P_j^i: \mathbb{R} \to [0,1]$, $1 \le j \le m, 1 \le i \le q$
P^i	acceptance probability of i-th DM, $i = 1, \dots, q$
P_i	probability that i-th objective function value is accepted by all DMs
P	probability that at least one solution in the portfolio S satisfies all DMs
m	number of objective functions
k	number of solutions in a portfolio
q	number of DMs

The proposed model uses a Bayesian framework, that is, probabilities are interpreted as quantified beliefs rather than being derived from frequencies. We will refer to the probability that the i-th DM will accept a solution as the *acceptance probability* $P^i(x)$. Here, we think of a large space X of n alternative solutions. A DM will surely reject a solution if her/his acceptance probability is 0. On the other hand, a DM will definitely accept a solution if his/her probability of acceptance is 1. Otherwise, the decision making is uncertain, and a DM's choice can be understood as a Bernoulli random variable, where the Bernoulli parameter is $P^i(x)$. The closer the probability is to 1, the higher the chance that this DM will accept the solution. A question that arises naturally is how we can determine the values of $P^i(x)$, $i = 1, \dots, q$. A method for doing so in the context of multicriteria optimization was recently proposed in [23]. It defines the acceptance probability based on probabilities of accepting values for objective functions by the DM. Therefore, it asks the DM/user to normalize the objective functions using the precise, yet intuitive, framework of desirability functions. Instead of stating a general formula for $P^i(x)$, for the i-th DM the decision behavior is modeled with respect to single objective functions. For instance, the DM is asked to specify a threshold for each objective, below which s/he would definitely reject a solution, and from which level on s/he gets indifferent to further improvements. The acceptance of solution with respect to each objective function would be guaranteed for values higher than this threshold. We will discuss this model in Section 2 in more detail. By aggregating this information, a belief probability distribution can be defined and processed in a Bayesian reasoning framework.

Given multiple, independent DMs we can for instance compute the probability that all q DMs accept a given solution as $P^1(x)P^2(x) \cdots P^q(x)$, and each of these probabilities is given as the joint probability over the events that the i-th DM will accept all values of the objective functions. In case of independent objective functions this would be $P(x) := \prod_{i=1}^{q} \prod_{j=1}^{m} P_j^i(x)$, where $P_j^i(x)$ denotes the probability that the i-th DM accepts the value of the j-th objective function. The maximization of this probability over all $x \in X$ can be accomplished by single objective optimization, and would yield an interesting single solution.

A more difficult question to answer arises, when sets of alternative solutions (here we will say: *portfolio*s of solutions) are considered and the goal is to find a set of k solutions (portfolio) with maximal probability that at least one of its elements satisfies all DMs. In practical decision making settings, due to cognitive limitations, it is common to consider only a small set of alternative solutions, so we may assume $k \ll |X|$. More precisely, we are interested in the following problem:

Problem: Given a set X with $n = |X|$ candidate solutions, m desirability functions, and q DMs: Find a subset of k solutions (portfolio) that maximizes the probability that at least one of its elements is accepted by all DMs.

For a single DM this question was discussed in [23]. In the following we will review this procedure, discuss computational aspects, and extend it to a procedure that can include multiple DMs. We will start the detailed discussion with some preliminaries on multiobjective decision analysis in Section 2. Then, in Section 3, we will discuss how to compute the acceptance probabilities based on desirability functions. In Section 4, the concept and computation of acceptance probabilities for portfolios is treated for a single DM. Section 5 extends the discussion to the scenario of multiple DMs. In Section 6, computational efficiency and exact procedures for optimal subset selection are examined. Finally, in Section 7 we summarize our findings and discuss future work.

2. Preliminaries

In this work we consider the *multiobjective optimization problem* (MOP) where the objective functions $f_1: X \to \mathbb{R}, \ldots, f_m: X \to \mathbb{R}$ are to be maximized for some search space $S \subseteq X$. It is said that $x \in S$ dominates y ∈ S, if and only if x is better or equal in all objective function values and strictly better in at least one objective function value. Pareto dominance in the objective space $\mathbb{R}^m$ is defined as: $y^{(1)} \in \mathbb{R}^m$ dominates $y^{(2)} \in \mathbb{R}^m$ if $y^{(1)} \neq y^{(2)}$ and $y^{(1)}$ is better or equal in all coordinates than $y^{(2)}$ [3]. For each goal vector $y \in \mathbb{R}^m$ let us define the attained subspace [20] as $DomSet(\{y\}) := \{u \in \mathbb{R}^m \mid y \text{ dominates } u\}$. For a finite multi-set (or population) of points $S = \{ y^{(1)}, \ldots, y^{(k)}\}$ of vectors in $\mathbb{R}^m$, define $DomSet(S) = DomSet(\{ y^{(1)}\}) \cup \ldots \cup DomSet(\{ y^{(n)}\})$. Furthermore, with $DomSets_m$ we denote the set of all attained subspaces A of $\mathbb{R}^m$ such that $A = DomSet(S)$ for some finite population $S \subset \mathbb{R}^m$. By a *set indicator* we mean here a real valued function that measures how well a population of k solutions performs. For instance, the *hypervolume indicator* is a well-studied set indicator in the context of multi-objective optimization. It is defined as: $HI_r(S) = Vol_m(DomSet(S) \cap [r, \infty])$ for $S \subset \mathbb{R}^m$. Here with Vol_m we denote the Lebesgue measure in dimension m (in 1-D length, in 2-D area, and in 3-D volume) and $r \in \mathbb{R}^m$ a user defined reference point.

A general way to construct set-indicators over $DomSets_m$ is by integrating over a density function $K: \mathbb{R}^m \to \mathbb{R}$ [2,12,20,23]. The density based hypervolume indicator function $DHI: DomSets_m \to \mathbb{R}$ is defined as:

$$DHI(S) := \int_{y \in DomSet(S)} K(y)\, dy.$$

Conditions for density functions that are compatible with the Pareto dominance relation were discussed in [23]. This definition is similar to the weighted hypervolume indicator of [2,12], but does not use a reference point. Probability density functions (PDFs) belong to the class of density functions that yield finite integrals and they are (weakly) compatible with Pareto dominance. The latter statement refers to the property that for two sets S and S', it holds that $DomSet(S) \supseteq DomSet(S') \Rightarrow DHI(S) \geq DHI(S')$. For probability density functions that are everywhere greater than zero, also strict monotonicity holds, that is $DomSet(S) \supset DomSet(S') \Rightarrow DHI(S) > DHI(S')$. In simple words, inserting additional points to the dominating set improves the set indicator. In the next Section, we will among other topics review why PDFs can be meaningful choices for densities.

To make this choice more explicit , we will write in the following $P(S) = DHI(S)$ for the acceptance probability for a set S extending the notion of acceptance probability for a singleton described in the previous section, and we will write $K(y) = PDF(y)$ for the kernel based on a probability density function.

3. Desirability Functions and Acceptance Probabilities

A user-friendly approach to construct probability density functions was described in [23]. The main idea is to derive acceptance probabilities from so called desirability functions, and take the derivatives† of those.

Desirability functions are widely used in multicriteria decision making in order to normalize values of objective functions[1,4,8]. Desirability functions, here for maximization, are functions $D_i: \mathbb{R} \to [0,1],\ i = 1, \dots, m$ that map objective function values to the interval $[0,1]$, where the value of 0 indicates that the solution is not acceptable, and a value of 1 indicates that the attained objective function value fully satisfies the DM and s/he would be indifferent to a further increase of the objective function value. All other values fall in the grey area.

There are two common types of desirability functions in the literature. Harrington desirability functions [5,8] and Derringer-Suich desirability functions [6].

The *Harrington* desirability functions are given by:

$$D_i(y_i) = \exp\big(-\exp\big(-(b_{0i} + b_{1i} y_i)\big)\big), i = 1, \dots, m.$$

Here, the constant b_0 defines the (almost) absolutely satisfying level and b_1 the marginally infeasible level. See Figure 1(a) for a visualization of shapes for different settings of b_0 and b_1. Harrington (1965) proposed the geometric mean $D_H(y) = \left(\prod_{i=1}^{m} D_i(y_i)\right)^{1/m}$ for computing a utility function for a solution called *the desirability index*. Here, we will use the m-th power of the geometric mean instead, because this gives rise to a more simple probabilistic interpretation: $D(y) := \prod_{i=1}^{m} D_i(y_i)$. The two aggregations are rank invariant to each other, that is, solutions will be ranked in the same way by them. Both forms have the property that a single desirability function value of zero causes the aggregate to be zero as well. However, the Harrington desirability functions obtain values in the open interval $(0,1)$ and therefore this property will only be relevant in the discussion of limit properties.

The *Derringer and Suich* desirability can well reach the boundaries of the interval $[0,1]$. They are defined by:

$$D_i(y_i) = \begin{cases} 0, if\ \ y_i < L_i \\ ((y_i - L_i)/(U_i - L_i))^{l_i} \quad if\ L_i \le y_i < U_i, \\ 1, if\ y_i \ge U_i \end{cases} \qquad i = 1, \dots, m$$

We use the same symbol D_i here as for the Harrington type, because in the following they will be used in a generic framework (in the subsequent expressions both can be applied). The input value below which the i-th desirability function has a value of 0 is named L_i and the input value above which the i-th desirability function has reached the value of 1 is named U_i. Moreover, l_i determines the *smoothness of the change* from low to high values. See Figure 1 (b) for a visualization of the different parameter settings.

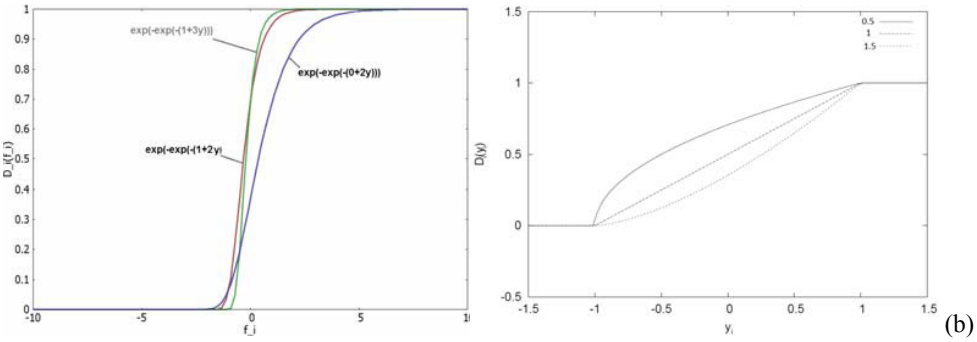
The fact that desirability functions for maximizing objectives can be seen as cumulative distribution functions [23] raises the question on whether there could be a probabilistic interpretation for them. Although the standard interpretation of desirability function is not probabilistic, a probabilistic interpretation of the desirability function could be as follows.

The desirability function $D_i(f_i(x))$ could be interpreted as the probability that the DM will accept a solution x if s/he would only look at the objective function f_i in isolation. The desirability index $D(f(x)) = \prod_{i=1}^{m} D_i(f_i(x))$ is the probability that a DM accepts all objective function values of x, assuming the probability variables for the acceptance of the single objective function values are independent of each other.

The choice of desirability functions for defining densities is discussed here, as their definitions can be accomplished in an intuitive way by a user. In essence only three parameters with a clear interpretation are required

† In the default case of independent objectives this approach is formally equivalent to an approach by Wagner and Trautmann [1], although the motivation and interpretation is different. See [23] for a discussion.

per objective function. Our approach will focus on the notion of acceptance probability using desirability functions, although any other cumulative density functions can in principle be used to model it, that is it can be used as $D_i(f_i(x))$. The important property of cumulative density functions with respect to maximization is that they are non-decreasing. This means whenever an objective function value gets higher the cumulative distribution function that measures the acceptance probability will stay equal or get higher as well.



(a) (b)

Figure 1 (a) Harrington type of desirability functions $\exp(-\exp(-(b_0 + b_1 x)))$ for different parameters of b_0 and b_1. (b) Derringer-Suich desirability functions for maximization. L is set to -1 and U is set to 1. l is set to 0.5, 1, and 1.5.

In addition, it should be remarked that so far we deliberately restricted the discussion to maximization in order to avoid confusion. However, it is simple and straightforward to reformulate minimization problems as maximization problems. For the three different types of problems, that are maximization, minimization, and distance to a target, the following transformations can be used:

Minimization	$P_i^j(y_i) := D_i^j(y_i)$
Maximization	$P_i^j(y_i) := D_i^j(-y_i)$
Minimize distance to a target T	$P_i^j(y_i) := D_i^j(-\lvert y_i - T \rvert)$

Note, that we use here the index $j \in \{1, \ldots, q\}$ for the DM, assuming that in general each DM can have a different desirability function.

4. Acceptance probabilities for portfolios for a single DM

In the previous section, it was discussed that $P^j(x) = \prod_{i=1}^{m} P_i\,(f_i(x))$ can be viewed as the acceptance probability for a single solution x by a single DM. In this section we will omit the upper index j, as it has always the value of 1 since we are dealing with the single DM case.

In [23], it was discussed how to extend this to compute set-based probabilities, in particular the probability that a set of solutions contains at least one acceptable solution for a single DM (here w.l.o.g. it has index 1). This probability was derived as a special case of a density-based hypervolume indicator of the i-th DM as

$$DHI_K(S) = \int_{y \,\epsilon\, DomSet(S)} PDF(y)\, dy$$

with

$$PDF(y) := (\partial^m P)/(\partial\, y_1 \ldots \partial y_m\,)\,(y)$$

and in the special case of a product form of P, as it is commonly used for independent acceptance probability functions:

$PDF(y)=(\partial P_1)/(\partial y_1) \cdot \ldots \cdot (\partial P_m)/(\partial y_m)$.

This was computed by reduction to the computation of the hypervolume indicator (with reference point 0) as in [23]. The transformation to the hypervolume indicator computation is described by the equation

$$P(S) = \int_{y \in \mathrm{DomSet(S)}} PDF(y)\,dy = HI_0(S') \text{ with } S' = \{(P_1, \ldots, P_m)(a) \mid a \in S\}.$$

The basic idea of the proof [23] is to use integration by parts over a partitioning of DomSet(S) into orthogonal ranges (cuboids) and for any decomposition into axis aligned cuboids of the integration regions the contributions of single cuboids are equivalent to the volume of cuboids in the transformed coordinate system. For the orthogonal range $[(l_1, \ldots l_m)^T, (u_1, \ldots, u_m)^T]$ one obtains

$$\begin{aligned}\int_{y_1=l_1}^{u_1} \ldots \int_{y_m=l_m}^{u_m} & PDF(y_1, \ldots, y_m)\, dy_1 \ldots dy_m \\ &= \int_{y_1=l_1}^{u_1} \ldots \int_{y_m}^{u_m} \prod_{i=1}^{m} \frac{\partial P_i}{\partial y_i} dy_1 \ldots dy_m \\ &= \prod_{i=1}^{m} (P_i(u_i) - P_i(l_i)) = Vol([(P_1(l_1), \ldots, P_m(l_m)), (P_1(u_1), \ldots, P_m(u_m))])\,.\end{aligned}$$

5. Multiple DMs

For multiple, say q, DMs the available information in the model will be the acceptance probability $P^i, i = 1, \ldots, q$ (for each DM), or, in case of m independent objective functions, the acceptance probabilities $P_j^i, j = 1, \ldots m$ (for each objective function and DM). The main idea for extension of the model to multiple DMs is to replace the probability that a single DM accepts a solution by the probability that all DMs find a solution acceptable. Assuming independent DMs, this corresponds to the joint cumulative density function of the individual $P^i, i = 1, \ldots, q$:

$$P(x) = \prod_{i=1}^{q} P^i(x)$$

Next, the density K can be derived from this joint cumulative distribution and based on this and the construction of density based hypervolume indicators (DHI) the performance indicator, which is the acceptance probability of a set $P(S)$:

$$P(S) := \int_{y \in DomSet(S)} PDF\,(y)\,dy\,, \text{ with } PDF\,(y) = \prod_{i=1}^{q} \frac{\partial^m P^i}{\partial y_1 \ldots \partial y_m}(y)$$

In case of independent objective functions we obtain:

$$K(y) = \prod_{i=1}^{q} \prod_{j=1}^{m} \frac{\partial P_j^i}{\partial y_j}(y)$$

Again we can reduce the computation of the integral using coordinate transformation to the hypervolume indicator:

$$P(S) = \int_{y \in \mathrm{DomSet(S)}} K(y)\,dy = HI_0(S') \text{ with } S' = \{(P_1, \ldots, P_m)(a) \mid a \in S\}$$

Here $P_j(a) = \prod_{i=1}^{q} P_j^i(a)$, which is the probability that all DMs are satisfied with the jth coordinate of a solution $a \in S$. The computational effort to compute this integral is fortunately only increased by a constant factor in the coordinate transformation as compared to that for a single DM.

A different view to motivate this approach would be to introduce a virtual entity called *moderator* as a global DM. This moderator finds a solution acceptable if and only if all DMs are satisfied. The moderator is a DM, too, but he has no independent opinion but just aggregates and follows the other DMs' preferences. The acceptance probability of the moderator for a solution with respect to the j-th objective function would be exactly given by $P_j(a) = \prod_{i=1}^{q} P_j^i(a)$.

6. Computational efficiency of indicator computation and optimal subset selection

An important question that arises is whether or not the approach can be implemented and how efficient the procedures are w.r.t. the governing parameters

- m (number of objectives),
- n (number of solutions in the search space X),
- k (number of solutions in the portfolio S),
- q (number of DMs).

Due to the reduction to the hypervolume indicator, the computational efficiency of the model equals to that of computations of the hypervolume indicator. The subset selection can be also performed in the transformed space and will yield the correct numbers. Fast algorithms for selecting optimal subsets are discussed in [21], [22]. The problem of selecting k points out of a set of n points such that the hypervolume indicator is maximized (over all sets of size k) is called the *hypervolume subset selection problem* (HSSP) in the literature [2].

Fast and exact selection for HSSP is currently only available for the 2-D case, where the best algorithms have a time complexity of $O(n(k + log\ n))$. For state of the art implementations of computing the hypervolume indicator we refer to [17] (2-D, 3-D case, time complexity is in $\Theta(k\ log\ k)$), and [7] (4-D case, $O(k^2)$). For the asymptotically fastest algorithm for N-D (N > 4) by Chan [18] there are to our best knowledge currently no implementations available. This recently discovered algorithm has a time complexity in $O(n^{m/3}\text{polylog}\ k)$. Due to the transformation we need to add a factor of $\Theta(q \cdot k \cdot m)$ to cover the pre-processing of the coordinates.

What does this mean in practice? In fact, if the number of objective functions is high we currently would have to use heuristic methods to compute optimal subsets. Already for 3 objectives this is a problem. Computing the indicator itself should be possible still for moderate numbers of objective functions (say up to four). The number of DMs is only influencing the $q \cdot k \cdot m$ – term. Therefore, the time complexity of computing the indicator as well as the subset selection problem will only linearly increase with the number of DMs. Note, that these are the only operations that require the computation of transcendental functions such as the *erf* function (this function is obtained by integrating the normal distribution), which is typically relatively time consuming as compared to computing elementary arithmetic operations.

7. Discussion and Outlook

The indicator that we have discussed in this paper measures the probability that the portfolio contains at least one single solution in the set that will be accepted by all DMs. It was shown, that by a nonlinear transformation of the coordinates (under independence assumptions) this indicator can be computed from information on acceptance distributions for single objective functions and DMs efficiently, if the number of objective functions is small (ca. 2-5). The problem of finding the subset of k solutions that maximizes this indicator among all such sets is however computationally more involved and only for the bi-objective case fast, polynomial time algorithms can so far be constructed. For other cases, we may find powerful heuristic algorithms for subset selection, such as genetic algorithms or tabu search.

A problem that remains unsolved is to take into account cases where consensus solutions are very unlikely or impossible. In this case one may still ask for a large number of DMs that should agree on a solution. An attractive goal could be to maximize the expected number of accepting DMs on some single solution in the portfolio.

Moreover, practical experiments and the discussion of correlation between the objectives and/or between the desirability functions provided by the DMs are topics for future research.

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